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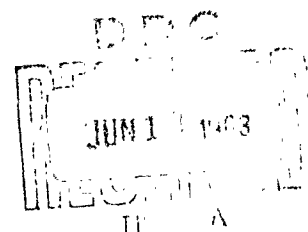
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THE DYNAMICS OF DISCONTINUITY SURFACES IN GENERAL RELATIVITY THEORY

Dominic G. B. Edelen and T. Y. Thomas

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The **RAND** Corporation
SANTA MONICA • CALIFORNIA

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PREFACE

Since the promulgation of the General Theory of Relativity, science has come to look upon gravitational phenomena as a direct consequence of the curvature of a four-dimensional space-time continuum arising from a momentum-energy distribution. While this viewpoint enables one to predict observed results with greater accuracy and is more satisfying in certain basic aspects than the Newtonian formulation, it requires solutions of a comparatively complicated system of nonlinear partial differential equations of the second order. In an attempt to shed some light on the properties of the solution manifold of the general relativity field equations with arbitrary momentum-energy tensors, we presented analyses of certain aspects of the general second-order discontinuity problem. The underlying idea behind this approach is that the analysis of discontinuity properties are usually considerably simpler than the analysis of the complete field equations—particularly if the field equations are nonlinear. Following the ideas delineated in our previous work concerning the evolutionary properties of the second-order problem, we present in this Memorandum a general dynamical theory of discontinuity surfaces and the associated jump strengths of both physical and geometrical quantities. The results reported here form the basis for a general analysis of galactic morphology which will be presented in succeeding communications.

SUMMARY

Let Σ be a time-like hypersurface in an Einstein-Riemann space E which is a support hypersurface for jumps in the momentum-energy tensor and in at least one second coordinate derivative of the metric tensor $h_{AB}(x)$. A system of necessary conditions is obtained for the existence of solutions of the Einstein field equations in the presence of jumps on Σ . These conditions are shown to be expressible in terms of surface tensors and surface tensorial differential systems. In particular, a system of algebraic surface tensor equations are obtained for the jump strengths of the metric tensor and a system of first order covariant surface differential equations are obtained for the jump strengths of the momentum-energy tensor. These equations involve only the surface components of the jump strengths and the components of the first fundamental form. An additional algebraic condition is obtained which involves the surface jump strengths of the momentum-energy tensor and the second fundamental form. If it is assumed that one knows the momentum-energy tensor on one side of Σ , explicit formulae are obtained for the continuation of the momentum-energy tensor across the discontinuity hypersurface Σ by integrating the differential system governing by the jump strengths of the momentum-energy tensor. This integration is effected by means of two principles

which state a fundamental relation between geometry and physics and the fact that physical systems are governed by second order differential relations. The case in which the hypersurface Σ forms a static three-dimensional hyperbolic-normal metric space Σ^* is then examined. Results are thus obtained which will be of significance in the application of the discontinuity method to problems in relativistic cosmology.

THE DYNAMICS OF DISCONTINUITY SURFACES IN GENERAL RELATIVITY THEORY

by

Dominic G. B. Edelen^{*}

and

T. Y. Thomas^{**}

1. INTRODUCTION

Since the promulgation of the General Theory of Relativity, science has come to look upon gravitational phenomena as a direct consequence of the curvature of a four-dimensional space-time continuum arising from a momentum-energy distribution. While this viewpoint enables one to predict observed results with greater accuracy and is more satisfying in certain basic aspects than the Newtonian formulation, it requires solutions of a comparatively complicated system of simultaneous nonlinear partial differential equations of the second order. Some success has been achieved in obtaining solutions, but the methods employed rely heavily on a large number of assumptions, and usually involve very restricted forms of the momentum-energy tensor or ignore this tensor altogether. With the exception of existence theorems for the initial-value problem, very few results have been obtained with respect to

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the properties of the solution manifold with arbitrary momentum-energy tensors.

In an attempt to shed some light on the properties of the solution manifold of the general relativity field equations with arbitrary momentum-energy tensors, we presented analyses of certain aspects of the general second-order discontinuity problem [1 through 6]. The underlying idea behind this approach is that the analysis of discontinuity properties are usually considerably simpler than the analysis of the complete field equations—particularly if the field equations are nonlinear. Such analyses also provide the basis upon which significant information concerning the detailed physical process can be gleaned; see, for example [7], [8] and [9]. Following the ideas delineated in our previous papers concerning the evolutionary properties of the second-order problem, we have obtained a general dynamical theory of discontinuity surfaces and the associated jump strengths of both physical and geometrical quantities. These results are presented here and will form the basis for a general analysis of galactic structures.

2. PRELIMINARY CONSIDERATIONS

In this section we collect certain required results from the differential geometry of hypersurfaces. We state these without proof and refer the reader to the standard texts or to the exposition given in [6].

Consider a metric space of the type used in general relativity theory; that is, an Einstein-Riemann space E whose metric structure is defined by the quadratic differential form

$$(2.1) \quad ds^2 = h_{AB} dx^A dx^B, \quad (A, B = 0, 1, 2, 3),$$

having signature -2 and coefficients h_{AB} which are functions of the coordinates x^A of the space E . As indicated in (2.1), capital Latin indices will have the range $0, 1, 2, 3$ and will be summed over this range in accordance with the summation convention. Let Σ denote a regular hypersurface in E ; that is, Σ can be defined parametrically by equations

$$(2.2) \quad x^A = f^A(u^0, u^1, u^2),$$

where the $f^A(u^\alpha)$ are continuously differentiable functions of the parametric or surface coordinates u^α such that the functional matrix $(\partial f^A / \partial u^\alpha)$ has rank 3 for all values of the u 's under consideration. Here and throughout this paper lower case Greek indices will be associated with surface quantities and obey the summation convention with the range $0, 1, 2$.

Over such a hypersurface Σ one can define a normal vector with covariant components N_A by the equations

$$(2.3) \quad x_\alpha^A N_A = 0 \quad (x_\alpha^A \stackrel{\text{def}}{=} \partial f^A(u) / \partial u^\alpha).$$

We shall assume throughout this paper that Σ is a time-like hypersurface, and hence its normal vector is space like. The normal vector to Σ may thus be normalized by the requirement

$$(2.4) \quad N_A N^A = -1.$$

The quantities x_α^A for $\alpha = 0, 1, 2$ are the components of three independent contravariant vectors in E at any point of Σ and can also be interpreted, for A fixed, as the components of four covariant vectors on the surface Σ .

The coefficients $a_{\alpha\beta}$ of the first fundamental form of the surface Σ are given, as functions of the surface coordinates u^α , by the equations

$$(2.5) \quad a_{\alpha\beta} = h_{AB} x_\alpha^A x_\beta^B.$$

As defined by (2.5), the quantities $a_{\alpha\beta}$ transform according to the tensor law when the surface coordinates undergo their admissible transformations. The surface tensor determined by the $a_{\alpha\beta}$ is called the metric tensor of the surface Σ . It can be shown that the first fundamental form of Σ is nonsingular and that it has signature -1. Also, denoting the inverse of $a_{\alpha\beta}$ by $a^{\alpha\beta}$, it can be shown that

$$(2.6) \quad a^{\alpha\beta} x_\alpha^A x_\beta^B = h^{AB} + N^A N^B.$$

Before proceeding further, it appears advisable to state the exact assumptions of continuity and differentiability which will be involved in this paper. For this purpose let us denote by J the region (open set) consisting of some neighborhood of E which contains Σ in its interior. Denote by J_1 the subregion of J lying on one side of Σ and by J_2 the subregion lying on the other side. Let us, furthermore, denote by D_1 the domain $J_1 + \Sigma$ (that is, the point set consisting of the subregion J_1 and the points of the surface Σ as boundary points), the domain $J_2 + \Sigma$ being denoted by D_2 . The following assumptions are now made.

A_1 : The functions $f^A(u^\alpha)$ in equations (2.2) defining the hypersurface Σ are of class C^3 .

A_2 : The metric components $h_{AB}(x^K)$ are functions of class C^1 in the region J and of class C^3 in the domains D_1 and D_2 .

It follows from these assumptions that the Christoffel symbols Λ_{BC}^A of the Einstein-Riemann space E are continuous across Σ . Also, from assumption A_1 and equations (2.6) it is seen that the metric components $a_{\alpha\beta}(u^\gamma)$ are continuous on Σ and have continuous first partial derivatives; hence the Christoffel symbols $\Lambda_{\beta\gamma}^\alpha$ determined by the quantities $a_{\alpha\beta}$ are continuous functions of the surface coordinates. One can thus construct the first surface covariant derivatives of differentiable tensorial quantities defined on Σ . Thus, in particular, we have

$$\begin{aligned} x^A_{\alpha;\beta} &= x^A_{\alpha,\beta} - x^A_\gamma \Lambda^\gamma_{\alpha\beta} + x^B_\alpha \Lambda^A_{BC} x^C_\beta, \\ N^A_{;\alpha} &= N^A_{,\alpha} + N^B \Lambda^A_{BC} x^C_\alpha \end{aligned}$$

for the components of the covariant derivatives of the mixed surface and space vectors defined on Σ by x^A_α and the unit normal vector N respectively. Use of the semicolon to denote covariant differentiation will be continued throughout this paper; correspondingly we shall use the comma to denote partial differentiation with respect to the space coordinates x^A or the surface coordinates u^α as indicated by the indices.

Denoting the coefficients of the second fundamental form on Σ by $b_{\alpha\beta}$ we have

$$b_{\alpha\beta} = -x^A_{\alpha;\beta} N_A.$$

Hence the $b_{\alpha\beta}$ are continuous functions of the surface coordinates and are moreover symmetric in the indices α and β since the quantities $x^A_{\alpha;\beta}$ are symmetric in these indices. The functions $b_{\alpha\beta}$ occur in the following important relations

$$(2.7) \quad x^A_{\alpha;\beta} = b_{\alpha\beta} N^A,$$

$$(2.8) \quad N^A_{;\alpha} = b_{\alpha\beta} a^{\beta\gamma} x^A_\gamma.$$

3. JUMP CONDITIONS

So far the hypersurface Σ has been any regular, time-like hypersurface in a four-dimensional Einstein-Riemann space E . We now restrict our attention to those hypersurfaces in E which carry basic field-theoretic information in the sense that they are the support hypersurfaces for the field discontinuities. In view of our previous assumptions, this is accomplished by the following requirement.

A_3 : There is a discontinuity in at least one of the second derivatives of the functions $h_{AB}(x^K)$ at points of the hypersurface Σ .

The symbol $|W_{B\dots}^{A\dots}|$ will be used to denote the jumps in the quantities $W_{B\dots}^{A\dots}$ across Σ ; that is, the differences in the limits of $W_{B\dots}^{A\dots}$ as points in Σ are approached from J_1 and J_2 respectively. By assumption A_2 , we have

$$(3.1) \quad |h_{AB}| = 0, \quad |h_{AB,C}| = 0,$$

while A_3 states that $|h_{AB,CD}| \neq 0$ for some choice of the indices. In fact, it can be shown [6] that there exist functions $\lambda_{AB}(u^\alpha)$ of the coordinates of the hypersurface Σ such that

$$(3.2) \quad |h_{AB,CD}| = \lambda_{AB} N_C N_D.$$

In view of (3.1) and the fact that

$$(3.3) \quad \lambda_{AB} = |h_{AB,CD}| N^C N^D,$$

the λ 's may be viewed as the jump strengths of the h -field.

We now impose the requirements that the structure of the space E is determined by the Einstein field equations

$$(3.4) \quad B_{AB} - \frac{1}{2} B h_{AB} = \kappa T_{AB}.$$

Here B_{AB} are the components of the Ricci tensor of E , B is the scalar curvature $B_{AB} h^{AB}$, T_{AB} are the components of the momentum-energy tensor, and κ is the so called gravitational constant. Let us denote the jump strengths of the momentum-energy tensor across Σ by S_{AB} ; that is

$$(3.5) \quad S_{AB} = |T_{AB}|,$$

where the S 's are functions of the surface coordinates u^α . It is then easily shown (see [2]) that a necessary condition for the existence of solutions to the Einstein field equations under assumptions A_1, A_2, A_3 is that the discontinuity strengths satisfy the equations

$$(3.6) \quad |B_{AB}| - \frac{1}{2} |B| h_{AB} = \kappa S_{AB}.$$

The system (3.6) also may be viewed as conditions for the continuation of solutions to the Einstein equations across surfaces of discontinuity of the momentum-energy tensor (see [4]).

Denoting the components of the complete curvature tensor of E by B^A_{BCD} , we have

$$B^A_{BCD} = \Lambda^A_{BC,D} - \Lambda^A_{BD,C} + *,$$

where the asterisk denotes terms which are quadratic in the Christoffel symbols. When use is made of these relations and equations (3.1) and (3.2) we obtain

$$(3.7) \quad 2|B^A_{BCD}| = h^{AM}(\lambda_{MC} N_B N_D + \lambda_{BD} N_M N_C - \lambda_{BC} N_M N_D - \lambda_{MD} N_B N_C),$$

from which the left-hand side of (3.6) may be evaluated. Introducing the quantities

$$(3.8) \quad \phi_A = \lambda_{AB} N^B, \quad \phi = \phi_A N^A,$$

the explicit evaluation of (3.6) is found to be given by

$$(3.9) \quad \lambda_{AB} + 2\phi_{(A} N_{B)} - \phi N_A N_B - (\phi + \lambda_{CD} h^{CD}) h_{AB} = 2\kappa S_{AB}.$$

If we multiply both sides of (3.9) by N^B , sum on the repeated index B , and use equations (3.8), the left members of the resulting equations vanish identically. We are thus left with the simple set of relations

$$(3.10) \quad S_{AB} N^B = 0.$$

It is also evident that the functions ϕ_A are undetermined by the equations (3.8), (3.9) and hence may be taken as arbitrary functions of the coordinates u^α on Σ . These functions clearly reflect the arbitrariness in the choice of coordinates of the space E . In addition, if there are no physical jumps (i.e., $S_{AB} = 0$), the admissible choice $\phi_A = 0$ annihilates the λ 's. In this sense, the metrical jump strengths determined by the ϕ 's have no intrinsic physical meaning.

The fundamental system of equations (3.9) and (3.10) involve the jump strengths λ_{AB} and S_{AB} , which are tensor quantities under admissible coordinate transformations of the four-dimensional space E . Now consider the quantities $S_{\alpha\beta}$ as defined by the equations

$$(3.11) \quad S_{\alpha\beta} = S_{AB} x_\alpha^A x_\beta^B.$$

Under admissible transformations of the surface coordinates u^α , these quantities transform according to the indicated tensor law, and hence constitute the components of a surface tensor. A direct calculation based on (2.6) and (3.10) gives

$$(3.12) \quad S^{AB} = S^{\alpha\beta} x_\alpha^A x_\beta^B,$$

where the $S^{\alpha\beta}$ are obtained from $S_{\alpha\beta}$ when we raise the indices by means of the contravariant quantities $a^{\alpha\beta}$ in the usual manner.

Equivalently, if we assume (3.12), equations (3.10) are identically satisfied. We thus see that the surface quantities $S_{\alpha\beta}$ give a unique determination of the components S_{AB} over any specified surface Σ , and that this determination identically satisfies equations (3.10).

One may also verify the following result:

$$(3.13) \quad S = S_{AB} h^{AB} = S_{AB} (x_{\alpha}^A x_{\beta}^B a^{\alpha\beta} - N^A N^B) \\ = (S_{AB} x_{\alpha}^A x_{\beta}^B) a^{\alpha\beta} = S_{\alpha\beta} a^{\alpha\beta}.$$

We now consider the surface tensor whose components are defined by the equations

$$(3.14) \quad \lambda_{\alpha\beta} = \lambda_{AB} x_{\alpha}^A x_{\beta}^B.$$

Let us write (3.9) in the equivalent form

$$(3.15) \quad \lambda_{AB} = 2 \kappa S_{AB} - 2 \phi ({}^A N_B) - (\phi + \kappa S) N_A N_B \\ - \kappa S h_{AB}$$

and contract both sides with $x_{\alpha}^A x_{\beta}^B$. The result is the following simple system of equations:

$$(3.16) \quad \lambda_{\alpha\beta} = \kappa (2 S_{\alpha\beta} - S a_{\alpha\beta}).$$

It is easily seen, when use is made of (3.12) and (3.16), that the relations (3.15) can be written in the equivalent form

$$(3.17) \quad \lambda^{AB} = 2(\lambda a^{\alpha\beta} - \lambda a^{\alpha\beta}) x_{\alpha}^A x_{\beta}^B - 2 \phi ({}^A N^B) \\ - (\phi - \lambda) N^A N^B + \lambda h^{AB},$$

where λ is defined by

$$(3.18) \quad \lambda = \lambda_{\alpha\beta} a^{\alpha\beta}.$$

Conversely the equations (3.17) in which the $\lambda^{\alpha\beta}$ are derived from (3.16) imply the relations (3.9) on Σ .

Combining the considerations of the previous paragraphs, we are thus led to the following basic result. Necessary conditions for the existence of solutions to the Einstein field equations under assumptions A_1, A_2, A_3 are given by the tensorial surface equations

$$(3.19) \quad \lambda_{\alpha\beta} = \kappa(2S_{\alpha\beta} - S a_{\alpha\beta}),$$

in which the surface quantities $S_{\alpha\beta}$ and $\lambda_{\alpha\beta}$ are determined from the jump strengths S_{AB} and λ_{AB} by (3.11) and (3.14) respectively.

By this result we are permitted to base our succeeding considerations on surface tensors and the intrinsic geometry of Σ .

4. DIFFERENTIAL RELATIONS

It is seen from the results of Sec. 3 that knowledge of the surface quantities $S_{\alpha\beta}$ is sufficient to determine the $\lambda_{\alpha\beta}$ and hence the λ_{AB} to within the geometrical structure of Σ , i.e. to within the functions $a_{\alpha\beta}$, N_A , x^A_α and h_{AB} on Σ . As yet, however, we have not used the full content of the Einstein theory for the relations

$$(4.1) \quad T^B_{A;B} = 0$$

are still at our disposal. We shall now show that the system (4.1) leads to a set of differential relations on Σ which will partially determine the structure of $S_{\alpha\beta}$ and the geometry of Σ .

We first note that from (3.5) we have

$$(4.2) \quad S_{A;\alpha}^B = |T_A^B|_{;\alpha} = |T_{A;C}^B| x_\alpha^C.$$

If we multiply (4.2) by $x_\beta^D a^{\alpha\beta}$, sum on the repeated index α and use (2.6), we are led to the equations

$$(4.3) \quad S_{A;\alpha}^B a^{\alpha\beta} x_\beta^D = |T_{A;C}^B| (h^{CD} + N^C N^D).$$

Obvious manipulations of (4.3) then give

$$(4.4) \quad |T_{A;C}^B| = S_{A;\alpha}^B a^{\alpha\beta} x_\beta^D h_{CD} - F_A^B N_C,$$

where

$$(4.5) \quad F_A^B = |T_{A;C}^B| N^C$$

are functions of the coordinates u^α of the hypersurface Σ . Now, since (4.1) holds on both sides of Σ , the tensor equations

$$(4.6) \quad |T_{A;B}^B| = 0$$

must hold on Σ . Substituting from (4.4) into (4.6), we then obtain the differential relations

$$(4.7) \quad S_{AB;\alpha} x_\beta^B a^{\alpha\beta} = F_A,$$

where the quantities F_A , which are given by

$$(4.8) \quad F_A = |T_{A;C}^B| N^C N_B,$$

constitute the components of a spacial vector on the hypersurface Σ .

We now replace the differential relations (4.7) by a set of equations which involve the quantities $S_{\alpha\beta}$ rather than the S_{AB} in accordance with the viewpoint expressed at the end of Sec. 3. For this purpose let us multiply both sides of (4.7) by N^A to obtain

$$(4.9) \quad N^A S_{AB;\alpha} x_\beta^B a^{\alpha\beta} = N^A F_A \stackrel{\text{def}}{=} \chi.$$

Then by covariant surface differentiation of (3.10) and use of (2.8) we find that

$$(4.10) \quad S^{\alpha\beta} b_{\alpha\beta} + \chi = 0.$$

On the other hand, if we contract both sides of (4.7) with x_γ^A , we have

$$(4.11) \quad S_{AB;\alpha} x_\beta^B x_\gamma^A a^{\alpha\beta} = F_A x_\gamma^A \stackrel{\text{def}}{=} F_\gamma.$$

It now follows from (2.7), (3.10) and (4.11) that

$$\begin{aligned} S_{\gamma;\alpha}^\alpha &= (S_{AB} x_\beta^B x_\gamma^A a^{\alpha\beta})_{;\alpha} = S_{AB;\alpha} x_\beta^B x_\gamma^A a^{\alpha\beta} + S_{AB} a^{\alpha\beta} (x_\beta^B x_\gamma^A)_{;\alpha} \\ &= F_\gamma + S_{AB} a^{\alpha\beta} (N^B b_{\alpha\beta} x_\gamma^A + x_\beta^B N^A b_{\alpha\gamma}) = F_\gamma. \end{aligned}$$

We have thus proved the following fundamental result. Necessary conditions for the existence of solutions to the Einstein field equations under assumptions A_1, A_2, A_3 are given by the relations

$$(4.12) \quad S^{\alpha\beta} b_{\alpha\beta} + \chi = 0,$$

$$(4.13) \quad S_{\beta;\alpha}^{\alpha} = F_{\beta},$$

over the surface Σ , where

$$(4.14) \quad \chi = F_A N^A, \quad F_{\beta} = F_A x_{\beta}^A, \quad F_A = |T_{A;C}^B| N^C N_B.$$

Although we will not pursue this approach, the above results may be used to obtain equations for the surface quantities $\lambda_{\alpha\beta}$.

To see this, we first solve (3.19) for $S_{\alpha\beta}$ and thereby obtain

$$(4.15) \quad 2 \kappa S_{\alpha\beta} = \lambda_{\alpha\beta} - \lambda a_{\alpha\beta}.$$

Substitution of (4.15) into (4.12) and (4.13) then leads to the desired results, namely

$$(4.16) \quad \lambda^{\alpha\beta} b_{\alpha\beta} - 2 \lambda \Omega + 2 \kappa \chi = 0,$$

$$(4.17) \quad \lambda_{\beta;\alpha}^{\alpha} - \lambda_{,\beta} = 2 \kappa F_{\beta},$$

where $\Omega = (1/2) b_{\alpha\beta} a^{\alpha\beta}$ is the mean curvature of Σ in E .

5. THE CONTINUATION PROBLEM

The results established in the previous section show that the jump strengths of the momentum-energy tensor and its covariant derivative cannot be specified in an arbitrary fashion in view of the conditions imposed by the Einstein field equations. However, a

certain degree of freedom is allowed. To see this, we first note that the F depends on the S 's and the geometry of Σ in accordance with the equations

$$(5.1) \quad F^A = S^{\alpha\beta} b_{\alpha\beta} N^A + S^{\alpha\beta}{}_{;\beta} x^A_\alpha.$$

This follows from (4.12), (4.13), (4.14) and the fact that (N^A) and (x^A_α) for $\alpha = 0, 1, 2$ form a vector basis on Σ . It is thus evident that if we specify $S_{\alpha\beta}$ in an arbitrary fashion and use our previous results, the quantities $\lambda_{\alpha\beta}$ and F_A are uniquely determined in a manner consistent with the Einstein field equations.

Suppose, now, that we know the momentum-energy tensor on one side of Σ and hence the limiting values of this tensor and its divergence as Σ is approached from that side. The question then arises as to the continuation of the momentum-energy tensor across Σ such that the Einstein field equations will be soluable. Since we have implicitly assumed that we do not know T_{AB} on the other side of Σ , we cannot use the field equations to determine the h 's and hence the λ 's. We can therefore not use (3.16) to determine the S 's since we do not know the λ 's. It thus follows that the only information at our disposal is provided by the existence conditions

$$(5.2) \quad S^{\beta}_{\alpha;\beta} = F_\alpha$$

and

$$(5.3) \quad S^{\alpha\beta} b_{\alpha\beta} + \chi = 0.$$

As previously noted, we could specify the S 's and then determine the quantities F^A by (5.1). This procedure would amount, however, to fitting the equations of motion (i.e., the divergence of the momentum-energy tensor) on one side of Σ to those on the other side by a specification of the quantities $|T_{A;C}^B| N^C N_B$, these latter quantities being uniquely determined by the arbitrarily chosen $S_{\alpha\beta}$. Conceptually, equations of motion are in some respects more fundamental than momentum-energy; in addition the quantities $|T_{A;C}^B| N^C N_B$ determine certain properties of the motion normal to Σ and hence are more easily observed than the S 's. Hence, having observed that such an artificial fitting process is possible, we shall disregard it as a general method of procedure. We are thus left with the problem of determining the S 's such that equations (5.2) and (5.3) are satisfied when the functions F_α and χ are arbitrarily pre-assigned functions of the coordinates u^α .

Let us first fix our attention on the system (5.2). These equations may be considered as an invariant differential system in the three-dimensional, hyperbolic-normal metric space Σ^* with coordinates u^α and metric differential form

$$(5.4) \quad d\sigma^2 = a_{\alpha\beta}(u^\gamma) du^\alpha du^\beta.$$

Because of the three-dimensional character of Σ^* , the components $K_{\alpha\beta\gamma\lambda}$ of its curvature tensor can be expressed by

$$K_{\alpha\beta\gamma\lambda} = E_{\alpha\beta\zeta} E_{\gamma\lambda\xi} (K^{\zeta\xi} - \frac{1}{2} K a^{\zeta\xi}),$$

where $E_{\alpha\beta\gamma}$ are the components of the permutation tensor of weight zero, $K_{\alpha\beta} = K^{\lambda}_{\alpha\beta\lambda}$, and $K = K_{\alpha\beta} a^{\alpha\beta}$. The Einstein tensor on Σ^* thus serves to determine the complete curvature tensor ($K_{\alpha\beta\gamma\delta}$).

If the system (5.2) is to possess a solution in Σ^* , there must exist a tensor $Q^{\alpha\beta}$ such that

$$(5.5) \quad Q^{\alpha\beta}_{;\beta} = F^{\alpha}, \quad (F^{\alpha} = F_{\beta} a^{\beta\alpha}).$$

In the context of differential equations, the tensor $Q^{\alpha\beta}$ plays the role of a particular solution of (5.2). The fact that (5.2) is a linear differential system in $S^{\alpha\beta}$ allows us to write the general solution in the form

$$(5.6) \quad S^{\alpha\beta} = Q^{\alpha\beta} + Z^{\alpha\beta},$$

where $Z^{\alpha\beta}$ is the general solution of the homogeneous system

$$(5.7) \quad Z^{\alpha\beta}_{;\beta} = 0.$$

To some extent it would appear that we have robbed Peter to pay Paul, for we now have to face the problem of determining the functions $Z^{\alpha\beta}$. Now, the system (5.7) is formally similar to the

equations of motion. In addition, the T 's represent the momentum-energy complex in E while the S 's represent part of the jump strength of momentum-energy and hence have energetic interpretations in Σ^* . The formal analogy goes even deeper, however, for the Z 's also partially determine the λ 's and these latter quantities represent the jumps in the second coordinate derivatives of the geometrical quantities h_{AB} . Also, as is well known, the intrinsic geometry of a discontinuity hypersurface enters into the determination of classical jump strengths in an essential fashion. It is thus natural to assume that the intrinsic geometry of Σ^* partially determine the S 's. Now, the quantities $Q^{\alpha\beta}$ are a particular solution of (5.2) and hence depend on the specific physical quantities F^α . Equations (5.6) thus show that the dependence of $S^{\alpha\beta}$ on the intrinsic geometry of Σ^* (that is, on functions of $K^\alpha_{\beta\gamma\lambda}$) must arise through the quantities $Z^{\alpha\beta}$. We are therefore led to state the following postulates for the integration of the system (5.7).

P_1 : The quantities $Z^{\alpha\beta}$ are the components of a metric tensor differential invariant of the space Σ^* .

P_2 : This metric tensor differential invariant is of the second order and linear in the second derivatives of the components of the metric tensor of Σ^* .

Postulate P_2 expresses the usual restriction to second order relations which is commonly assumed in physical theories. On the

other hand, postulate P_1 together with (5.6) is a fundamental statement concerning the relation between physics and geometry.

The procedure is now straightforward. It follows from (5.7) that the divergence of the metric tensor differential invariant $(Z^{\alpha\beta})$ must vanish identically. From the known procedures for constructing such tensor differential invariants [10], it follows that the most general quantity which satisfies our requirements is given by

$$(5.9) \quad Z_{\alpha\beta} = \Theta (K_{\alpha\beta} - \frac{1}{2}(K + \Pi)a_{\alpha\beta}),$$

where Θ and Π are constants. We shall henceforth assume that these constants have fixed values, and in particular that $\Theta \neq 0$.

We have now established a consistent procedure whereby S 's may be obtained such that we simultaneously satisfy the existence requirements (5.2), for (F_α) an arbitrarily assigned surface vector, and the requirements of postulates P_1 and P_2 . We still have to satisfy the existence requirement (5.3). Now, the Q 's are a particular solution of the linear system (5.2) and hence we may write

$$(5.10) \quad \alpha_{\alpha\beta} = W_{\alpha\beta}(F_\gamma) + P_{\alpha\beta},$$

where the W 's are unique functionals of the F 's such that

$$(5.11) \quad \lambda W_{\alpha\beta}(F^\gamma) = W_{\alpha\beta}(\lambda F_\gamma)$$

is an identity in λ and the P 's are any functions of the coordinates u^α for which

$$(5.12) \quad P^{\alpha\beta}_{;\beta} = 0.$$

If we substitute (5.6) and (5.10) into (5.3), we obtain

$$(5.13) \quad P^{\alpha\beta} b_{\alpha\beta} = -\chi - (W^{\alpha\beta} + Z^{\alpha\beta}) b_{\alpha\beta}.$$

Since all of the terms on the right-hand side of this equation are known, (5.13) is seen to be an equation for the determination of $P_{\alpha\beta}$. Combining this with (5.12), we have four equations for the determination of the six quantities $P_{\alpha\beta}$. Hence the existence requirements (5.3) can always be satisfied; in fact, we are free to add two more conditions in most cases.

Combining the above results, we have the following conclusion. Let the functions χ and F^α be arbitrarily assigned functions of the coordinates u^α . Then a consistent procedure for the continuation of T_{AB} across Σ , such that (1) Postulates P_1 and P_2 are satisfied and (2) the Einstein field equations are soluble, is given by

$$(5.14) \quad S_{\alpha\beta} = Q_{\alpha\beta} + \Theta (K_{\alpha\beta} - \frac{1}{2} (K + \Pi) a_{\alpha\beta}),$$

$$(5.15) \quad K^{\alpha\beta} b_{\alpha\beta} - \Theta (K + \Pi) \Omega + Q^{\alpha\beta} b_{\alpha\beta} + \chi = 0,$$

$$(5.16) \quad \lambda_{\alpha\beta} = 2\kappa (\Theta K_{\alpha\beta} + Q_{\alpha\beta}) - \frac{\kappa}{2} (\Theta (K - \Pi) + 2Q) a_{\alpha\beta},$$

$$(5.17) \quad Q^{\alpha\beta}_{;\beta} = F^\alpha,$$

where Θ and Π are constants.

6. THE DISCONTINUITY HYPERSURFACE AS A STATIC 3-SURFACE

The problem of continuing the momentum-energy tensor across Σ arose because of an assumed lack of knowledge of T_{AB} on one side of Σ . There are a number of important physical problems in which this is the case and, in addition, the functions $h_{AB}(x^K)$ are unknown—either because the Einstein field equations have not been solved or because there is insufficient information to effect such a solution. Fortunately, this lack of information is usually compensated by known properties of the discontinuity hypersurface. We shall now consider a particularly important example of this situation.

Let Σ be a hypersurface for which the three-dimensional metric space Σ^* is known to be static; that is, the space Σ^* admits a one-parameter group of isometries whose trajectories form a time-like normal congruence [11]. This means that there is a vector field Y defined on Σ^* such that

$$(6.1) \quad Y_{(\alpha;\beta)} = 0, \quad Y_{[\alpha} Y_{\beta}; \gamma] = 0, \quad Y_{\alpha} Y^{\alpha} = e^{2\Psi},$$

where Ψ is a finite-valued function of the coordinates u^{α} . From the physical point of view, these conditions state that the infinitesimal 2-spaces of observers traveling along the curves of the congruence (i.e., the curves defined by $du^{\alpha}/dp = Y^{\alpha}(u^{\beta})$ where p is the parameter on the congruence) mesh into finite, space-like

two-surfaces called space sections. These space sections are mapped isometrically onto each other by the group of point transformations generated by the congruence. The space sections of Σ^* , and hence of Σ , correspond to what we would normally refer to as the two-dimensional boundary of a three-dimensional space-like body obtained from E by an appropriate time-section. In addition, the vector field with components $Y^A = Y^\alpha x_\alpha^A$ in E is everywhere tangent to Σ and transports the first fundamental form on Σ without change. Hence, the requirement that Σ^* be static is equivalent to the requirement that Σ be generated by the two-dimensional boundary \mathcal{B} of a body acted upon by a one-parameter group of point transformations which preserves the metric structure on \mathcal{B} ; that is \mathcal{B} is metrically stable in E .

It should be noted that the requirement that Σ^* be static does not necessarily imply that the Einstein-Riemann space E is static. For instance, suppose that there is a C^1 vector field X_A defined in E and such that $\bar{X}_A x_\alpha^A = Y_\alpha$, where the bar denotes evaluation on Σ . We then have

$$\begin{aligned} Y_{(\alpha;\beta)} &= (\bar{X}_A x_{(\alpha)}^A)_{;\beta} = \bar{X}_{A;(\beta} x_{\alpha)}^A + \bar{X}_A N^A_{\beta}{}^{\alpha} \\ &= \bar{X}_{(A;B)} x_\alpha^A x_\beta^B + \bar{X}_A N^A_{\beta}{}^{\alpha} . \end{aligned}$$

Hence, if Σ^* is static, $\bar{X}_{(A;B)}$ can vanish for $b_{\alpha\beta} \neq 0$ only if $\bar{X}_A N^A$ vanishes.

In order to obtain the implications of the system (6.1), we define a vector U in Σ^* by the equations

$$(6.2) \quad U_{\alpha} = e^{-\Psi} Y_{\alpha}.$$

Using (6.2) to eliminate the vector Y from the system (6.1), we obtain the equivalent system (see [12], pp. 65 f)

$$(6.3) \quad U_{\alpha} U^{\alpha} = 1, \quad U_{\alpha;\beta} = \dot{U}_{\alpha} U_{\beta}, \quad \dot{U}_{\alpha} = U_{\alpha;\beta} U^{\beta} = \Psi_{,\alpha}.$$

As a direct consequence of this system we have

$$(6.4) \quad 2 U_{\alpha;[\beta\gamma]} = K_{\alpha\beta\gamma} U^{\sigma} = 2 \dot{U}_{\alpha;[\gamma} U_{\beta]} + 2 \dot{U}_{\alpha} \dot{U}_{[\beta} U_{\gamma]}.$$

Hence, if we contract (6.4) on the indices (α, γ) and note that (6.3) implies

$$\dot{U}_{\alpha;\beta} U^{\alpha} = -\dot{U}^{\alpha} \dot{U}_{\alpha} U_{\beta},$$

we finally obtain the relations,

$$(6.5) \quad K_{\alpha\beta} U^{\sigma} = (a^{\alpha\gamma} \Psi_{;\alpha\gamma}) U_{\beta}.$$

The content of equations (6.5) is that the Ricci tensor of Σ^* admits U as a time-like eigenvector and that the corresponding eigenvalue is the D'Alembertian of Ψ , i.e., $a^{\alpha\beta} \Psi_{;\alpha\beta}$.

We now apply these conditions to the continuation problem for

T_{AB} . If we contract (5.14), we have

$$(6.6) \quad \Theta K = -2(S - Q + \frac{3}{2} \Theta \Pi),$$

where we have used the obvious notation $S = S_{\alpha\beta} a^{\alpha\beta}$ and $Q = Q_{\alpha\beta} a^{\alpha\beta}$. We may then solve (5.14) for $K_{\alpha\beta}$ to obtain

$$(6.7) \quad \Theta K_{\alpha\beta} = S_{\alpha\beta} - Q_{\alpha\beta} - (S - Q + \Theta \Pi) a_{\alpha\beta}.$$

Using (6.7) and the fact that $K_{\alpha\beta}$ has to satisfy the conditions stated by (6.5) if Σ^* is static, we can obtain the corresponding requirements which $S_{\alpha\beta}$ must satisfy. These requirements are

$$(6.8) \quad (S_{\alpha\beta} - Q_{\alpha\beta}) U^\beta = \rho U_\alpha,$$

for ρ given by

$$(6.9) \quad \rho = \Theta a^{\alpha\beta} \Psi_{;\alpha\beta} + S - Q + \Theta \Pi.$$

Equation (6.8) states that $S_{\alpha\beta} - Q_{\alpha\beta}$ admits U as a time-like eigenvector with associated eigenvalue ρ ; assuming that this eigenvalue is simple, we obtain a representation for $S_{\alpha\beta}$, namely

$$(6.10) \quad S_{\alpha\beta} = Q_{\alpha\beta} + \rho U_\alpha U_\beta + \sigma_{\alpha\beta}.$$

The tensor with components $\sigma_{\alpha\beta}$ which appears in (6.10) is arbitrary to within the requirements that it be symmetric and admit

U as a null vector. The representation (6.10) gives

$$(6.11) \quad S = Q + \rho + \sigma_{\alpha}^{\alpha},$$

and hence (6.9) leads to the interesting result.

$$(6.12) \quad \Theta (a^{\alpha\beta} \Psi_{;\alpha\beta} + \Pi) + \sigma_{\alpha}^{\alpha} = 0.$$

Since we have exact requirements of the U 's, namely equations (6.3), we may obtain significantly more information concerning the structure of $S_{\alpha\beta}$ for static spaces Σ^* . Without loss of generality, we may assume that

$$(6.13) \quad S_{\alpha\beta} = \mu U_{\alpha} U_{\beta} + Q_{\alpha\beta} + M_{\alpha\beta},$$

where the quantities μ and $M_{\alpha\beta}$ are to be determined so that U is an eigenvector of M . Noting that (6.3) implies

$$(6.14) \quad \Psi_{,\alpha} U^{\alpha} = 0, \quad U^{\alpha}_{;\alpha} = 0,$$

the substitution of (6.13) into the equations (5.2) leads to

$$(6.15) \quad \dot{\mu} U_{\alpha} + \mu \dot{U}_{\alpha} + M^{\beta}_{\alpha;\beta} U^{\alpha} = 0,$$

and hence

$$\dot{\mu} + M^{\beta}_{\alpha;\beta} U^{\alpha} = 0.$$

If we use the fact that $\dot{U}_{\alpha} = \Psi_{,\alpha}$ and then add and subtract the

quantity $(\mu \Psi)_{,\alpha}$, the system (6.15) becomes

$$(6.16) \quad (U_{\alpha} U^{\beta} - \Psi \delta_{\alpha}^{\beta})_{\mu, \beta} + (M_{\alpha}^{\beta} + \mu \Psi \delta_{\alpha}^{\beta})_{;\beta} = 0.$$

Thus, since

$$(6.17) \quad \sigma_{\alpha\beta} = M_{\gamma\eta} (\delta_{\alpha}^{\gamma} - U_{\alpha} U^{\gamma}) (\delta_{\beta}^{\eta} - U_{\beta} U^{\eta}),$$

and

$$(6.18) \quad M_{\alpha}^{\beta} U^{\alpha} = \xi U^{\beta}$$

we have the seven equations (6.12), (6.17) and (6.18) for the determination of the eight functions ξ, μ , and $M_{\alpha\beta}$.

As an indication of the use of these results, consider the particular case in which the function μ is constant. The system (6.16) then reduces to

$$(6.19) \quad (M^{\alpha\beta} + \Psi \mu a^{\alpha\beta})_{;\beta} = 0,$$

and hence it is possible to consider the particular situation in which we have

$$(6.20) \quad M_{\alpha\beta} = -\mu \Psi a_{\alpha\beta}.$$

The equations (6.13) and (6.17) then give

$$(6.21) \quad S_{\alpha\beta} = Q_{\alpha\beta} + \mu (U_{\alpha} U_{\beta} - \Psi a_{\alpha\beta}),$$

and

$$(6.22) \quad \sigma_{\alpha\beta} = -\mu \Psi (a_{\alpha\beta} - U_{\alpha} U_{\beta}).$$

Hence, substituting (6.22) into (6.17) we obtain the important result

$$(6.23) \quad \Theta(a^{\alpha\beta} \Psi_{;\alpha\beta} + \Pi) - 2 \mu \Psi = 0.$$

The function Ψ thus satisfies a Schroedinger type equation, and in addition, must be such that

$$(6.24) \quad \Psi_{;\alpha} U^{\alpha} = 0.$$

This last requirement follows from (6.14).

Although the above example is highly artificial in the manner in which it has been introduced here, related situations arise in certain fundamental problems associated with relativistic cosmology and galactic structure. These problems will be treated in succeeding papers in which equations similar to (6.23) lead to new and fundamental results.

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